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Some gas flows which obey Charles' Law

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The effects of substantial temperature and hence density changes on a low-speed 'incompressible' flow can be modelled by adopting Charles' Law as one of the equations of state. It is found that planar radial inflows or outflows constitute a group of solutions which are self-similar for arbitrary temperature variations. When the temperature is written as a separable function of radius and polar angle, ordinary differential equations result. Permissible solutions include some with discontinuities in the temperature gradient across a radial line (streamline); this is a rough model of a diffusion flame and it is used to illustrate some features of a variable-density flow in a channel with radial walls in the presence of such a 'flame'.

Exact analytical solutions are given for the situation in which temperature increases linearly with radius; no boundary layers appear for either outflow or inflow. Approximate analytical solutions are presented for the case of a relatively rapid inflow with temperature independent of the radius; a velocity boundary layer exists at the walls and in the neighbourhood of the 'flame', although the latter is of small velocity amplitude.

1. Introduction

There are many situations, especially in the field of combustion studies, which involve a gas flow that is slow enough to make compressibility effects negligible, but which, nonetheless, involve substantial density variations as a result of some kind of heating. In such circumstances it is permissible to simplify the familiar equation of state which links pressure p, density ρ, temperature T and molecular weight W by adopting Charles' Law. With W constant, this results in the elementary statement that the product ρT is fixed and this is the form that we shall adopt in the present work.

It is consistent to neglect the pressure-work and viscous dissipation terms in the energy equation but, apart from these simplifications, the Navier–Stokes equations are used in full and some elementary solutions are sought through the medium of a similitude. The details can be found in the following sections, but we remark here that the solutions constitute an extension of the class of Jeffery–Hamel flows of a constant-density viscous fluid in a diverging channel with plane walls (Jeffery 1915; Hamel 1917), and of their compressible-flow analogue due to Wilhelm (1973). The present paper deals only with analytical matters; an exact solution for both inflow and outflow of a constant-viscosity gas is exhibited in § 5 and a boundary-layer approximation for the inflow of a gas whose viscosity varies with temperature occupies § 6.

2. The equations

The equations which describe the planar steady motion of a fluid which obeys Charles' Law

$$\rho T = \rho_0 T_0 \tag{1}$$

are

$$(\rho U)_x + (\rho V)_y = 0, \tag{2}$$

$$\rho\{UU_x + VU_y\} = -\{p + \tfrac{2}{3}\mu[U_x + V_y]\}_x + (2\mu U_x)_x + \{\mu(V_x + U_y)\}_y, \tag{3}$$

$$\rho\{UV_x + VV_y\} = -\{p + \tfrac{2}{3}\mu[U_x + V_y]\}_y + (2\mu V_y)_y + \{\mu(V_x + U_y)\}_x, \tag{4}$$

$$\rho C_p\{UT_x + VT_y\} = (\lambda T_x)_x + (\lambda T_y)_y. \tag{5}$$

The x and y velocity components are U and V; μ is the dynamic viscosity coefficient, λ is the thermal conductivity and C_p, the specific heat at constant pressure, has been assumed to be constant; T is the absolute temperature, ρ is the density and p is the pressure.

The Mach number is assumed to be low enough for pressure-work and viscous dissipation terms to be negligible in the energy equation (5). This is consistent with the adoption of (1) and it is evident from that equation that the variables with a subscript zero constitute an appropriate set of 'background' or 'basic' values. Equations (1)–(5) will enable us to study the effects of significant temperature changes in an essentially low-speed motion. In particular, we remark on the coupling which exists between U and V variations and T variations, especially via (1) and the continuity relation (2).

3. Similitude

The problem presented by the solution of (1)–(5) is greatly simplified if similarity groupings of the variables can be found. Following the standard procedure as set out by Birkhoff (1960, chap. 4), it is possible to show that such a similitude is given by

$$\rho U(x \,\mathrm{or}\, y) = \mu f(y/x), \quad \rho V(x \,\mathrm{or}\, y) = \mu g(y/x), \quad p = \rho(U^2 \,\mathrm{or}\, V^2)\, h(y/x).$$

Equations (1)–(5) impose no direct limitation on the behaviour of the temperature T. The forms just quoted are clumsy and not especially useful but if we change to a polar co-ordinate system

$$\left.\begin{aligned} \theta = \tan^{-1}(y/x), \quad r = (x^2+y^2)^{\frac{1}{2}}, \\ u = U\cos\theta + V\sin\theta, \quad v = -U\sin\theta + V\cos\theta, \end{aligned}\right\} \tag{6}$$

it is readily seen that the similitude represents a purely radial flow (i.e. V/U is a function of y/x, or θ, only). Thus v is zero and the radial velocity component u and pressure p must be derived from the relations

$$\rho u r = \mu F(\theta), \quad p\rho r^2 = \mu^2 P(\theta). \tag{7}$$

Substituting the similarity form into the continuity equation (2) shows that

$$r^{-1}\mu F(\theta)\, \partial\mu/\partial r = 0,$$

so that the similitude can be consistent with this conservation requirement only if

$$\text{(I)}\ \mu = \text{constant} \tag{8}$$

or

$$\text{(II)}\ \mu = \mu(\theta). \tag{9}$$

The similitude also requires λ/μ to be an invariant ratio, whence it follows that λ must obey restrictions similar to those in (8) and (9).

However, the Prandtl number

$$Pr = \mu C_p/\lambda \tag{10}$$

is a constant for all practical purposes, so that no additional limitation is imposed by λ if we choose to deal with the situation of constant Prandtl number. In dealing with gases it is of course necessary to recognize that μ and λ will, in reality, vary with T. Indeed $\mu \propto T$ is often a good approximation and it is clear that if we take account of density variations by using Charles' Law (1) then we should also admit temperature dependence of μ and λ. Adopting the relation

$$\mu/\mu_0 = (T/T_0)^\beta, \quad \beta \geqslant 0, \tag{11}$$

it is clear that for any β other than zero condition (9) imposes the similitude

$$\text{(II) } T = T_0 H(\theta) \tag{12}$$

on the temperature.

Case II is then practical but rather restrictive, so that we shall devote some time to the rather more artificial situation of case I, for which $\beta = 0$. We may excuse this on grounds of expediency, since it is the primary T, ρ, u connexion that we wish to explore, and any variations of μ with T can perhaps be rationalized as in some measure secondary effects. The r, θ dependence of T is unrestrained in case I, which leaves us with simultaneous partial differential equations to solve, but if we choose to make T a separable function of the r, θ co-ordinates, it is readily shown that the only separable form of the energy equation occurs when

$$\text{(I) } T = T_0(r/r_0)^n H(\theta). \tag{13}$$

This definition makes $H(\theta)$ dimensionless, as in (12); r_0 is evidently any convenient reference value of the radial co-ordinate. We shall adopt (13) from now on for case I. The equations satisfied by the functions $F(\theta)$, $P(\theta)$ and $H(\theta)$ can be found by substituting (7)–(13) into the momentum and energy equations (3)–(5), remembering that v is zero. The results for cases I and II are summarized below; a prime indicates differentiation with respect to θ.

Case I (μ = constant)

$$(n-1)\,HF^2 = -(n-2)\,HP + \tfrac{4}{3}n(n-2)\,HF + (HF)'', \tag{14a}$$

$$0 = -(HP)' + (\tfrac{1}{3}n+2)\,(HF)', \tag{14b}$$

$$nPrFH = n^2H + H''. \tag{14c}$$

Equations (14a, b) combine to give

$$(n-1)\,(HF^2)' = (n-2)^2\,(HF)' + (HF)'''. \tag{14d}$$

Case II ($\mu \propto T^\beta$, $\beta \geqslant 0$)

$$-\mu^2F^2H = 2\mu^2HP + \mu'(\mu FH)' + \mu(\mu FH)'', \tag{15a}$$

$$0 = -(\mu^2HP)' + 2\mu'(\mu FH) + 2\mu(\mu FH)', \tag{15b}$$

$$0 = (\mu H')'. \tag{15c}$$

Equations (15a, b) combine to give

$$\{[\mu(\mu FH)']' + 4\mu(\mu FH) + (\mu F)^2H\}' = 0. \tag{15d}$$

We observe that (1), (7) and (13) combine to give

$$u = \left(\frac{\mu}{\rho_0 r_0}\right)\left(\frac{r}{r_0}\right)^{n-1} FH, \quad p = \left(\frac{\mu}{\rho_0 r_0}\right)^2 \left(\frac{r}{r_0}\right)^{n-2} HP, \tag{16}$$

so that the gas velocity depends upon the product μFH and the pressure depends upon the product $\mu^2 HP$.

4. Symmetrical channel flows

The similarity equations of the previous section describe behaviour in the special case of constant temperature ($n = 0$, $H = 1$), when it is easily seen that either (14a) or (15a) reduces to the classical form of equation that has been thoroughly investigated in the past. A concise account can be found in Whitham's article (Rosenhead 1963, pp. 142–150) and we observe that in this constant-temperature, and hence constant-density, situation the presence of two channel walls at $\theta = \pm\alpha$ (say) imposes two conditions on F which are expressions of the no-slip requirement at a solid surface. Specification of

$$F(0) \equiv F_0 \tag{17}$$

is then sufficient to determine the solution when the temperature and density are constant.

In the present case two further conditions are generally ($n \neq 0$ and/or $H \neq$ constant) required for the temperature function H and these may well be provided by the specification of the wall temperatures. However, there are other situations of interest and we shall limit the discussion here by first imposing an overall symmetry on the system. With specification of the centre-line temperature we then consider two cases; in the first we assume that the temperature gradient with θ is continuous, and hence zero, and in the second case we assume that $H'(0+) = -H'(0-)$ is non-zero, negative and given. The latter situation is a rough model of a diffusion flame lying along the channel's centre-line and it is necessary to note that the essential continuity of the rate-of-strain tensor in the neighbourhood of such a flame (Clarke 1967) makes it essential to have $(\mu FH)'$ equal to zero when $\theta = 0$. The boundary conditions chosen for the present work are summarized as cases A and B:

Case A

$$F(\alpha) = 0, \quad F(0) = F_0, \quad F'(0) = 0, \quad H(0) = 1, \quad H'(0) = 0. \tag{18}$$

Case B

$$F(\alpha) = 0, \quad F(0) = F_0, \quad F'(0) = F_0 H_0', \quad H(0) = 1, \quad H'(0) = -H_0'. \tag{19}$$

The definition of $F(\theta)$ in (7) shows that it is actually a local Reynolds number based on the radius r; specification of F_0 as in (17) is therefore equivalent to selection of the centre-line value of this Reynolds number. It is sometimes more convenient, or even more realistic, to classify a channel flow by the total mass flux through it and we observe, again from (7), that this quantity is given by the integral

$$Q = \left|\int_{-\alpha}^{\alpha} \mu(\theta) F(\theta)\, d\theta\right|. \tag{20}$$

The modulus of the integral is used in (20) to make Q positive for both net outflow and net inflow; the connexion between Q and F_0 is evidently not direct.

5. Analytical results for $n = 1$

When μ is constant and $n = 1$ equations (14*c*, *d*) become

$$(HF)''' + (HF)' = 0, \tag{21}$$

$$H'' + H = PrFH, \tag{22}$$

and this pair of equations clearly has simple solutions in terms of trigonometric functions.

In particular, it is found that

$$HF = F_0(\cos\theta - \cos\alpha)/(1-\cos\alpha) \tag{23}$$

for both case A and case B boundary conditions. When α, and hence θ, is small,

$$HF \simeq F_0\{1-(\theta/\alpha)^2\} \tag{24}$$

and (16) and (24) show that the *velocity* distribution is approximately parabolic in θ and exactly independent of the radius.

Turning to the temperature distribution, (22) has a solution for case A boundary conditions which can be written as

$$H = H_A(\theta) \equiv \cos\theta - PrF_0\cos\alpha\left\{\frac{1-\cos\theta}{1-\cos\alpha}\right\} + \tfrac{1}{2}PrF_0\left\{\frac{\theta\sin\theta}{1-\cos\alpha}\right\}. \tag{25}$$

This temperature distribution has a stationary value on the centre-line of the channel, by hypothesis [see (18)], but we observe that

$$H_A''(0) = PrF_0 - 1, \tag{26}$$

so that this stationary value is a minimum (maximum) for $PrF_0 > 1$ (< 1).

If α, the half-angle of the channel, is confined to the range $0 < \alpha < \frac{1}{2}\pi$ there can be at most one more stationary value for H_A within each half of the channel, and this stationary value must be a maximum. The existence of these two temperature maxima, symmetrically disposed on either side of the centre-line temperature minimum, is only possible for a limited range of PrF_0 for any given α. In particular, it is necessary to have $PrF_0 > 1$ and this product must also be less than the value $(PrF_0)_{max}$, where

$$2\{1-1/(PrF_0)_{max}\}\{1-\cos\alpha\} = 1-\alpha/\tan\alpha. \tag{27}$$

When PrF_0 is equal to $(PrF_0)_{max}$, the temperature gradient with respect to θ falls to zero at the walls, which therefore conduct no heat into or out of the channel. If PrF_0 exceeds $(PrF_0)_{max}$, the maximum H_A values lie outside the channel and are therefore of no physical significance.

It must be remembered that the temperature T is equal to $T_0(r/r_0)H(\theta)$, so that the present configuration is essentially one for which the gas flows between walls which grow hotter with increasing radius. The slightly unexpected behaviour of T described above can then be understood in terms of the competing influences of convection and conduction. When convection is strong and in the outflow (r-increasing) direction it is necessary to conduct energy into the gas through the walls in order to maintain the motion; the convection requirement here is $PrF_0 > (PrF_0)_{max} > 0$. When PrF_0 is less than $(PrF_0)_{max}$ energy must be *removed* through the walls for the same purpose.

When PrF_0 is small enough, certainly for $PrF_0 < 1$ and including negative values of F_0, the wall temperature

$$T_w = T_0(r/r_0)\,H(\alpha) \equiv T_0(r/r_0)\,H_w \tag{28}$$

will be less than the centre-line value $T_0(r/r_0)$. It is necessary for the sake of physical reality to ensure that T_w, which is the minimum value of the temperature within the channel in the circumstances, shall not become negative [recall that T is essentially an absolute temperature; see (1)]. The criterion is evidently $H_w > 0$, and this readily translates into

$$PrF_0 > -[-1+\alpha\tan\alpha/2(1-\cos\alpha)]^{-1} \tag{29}$$

when $0 < \alpha < \frac{1}{2}\pi$. The right-hand side of (29) is negative for $0 < \alpha < \frac{1}{2}\pi$, so that the inequality is satisfied for all positive (outflow) values of F_0. If F_0 is negative (29) may not be satisfied. For example, if α is small (29) requires

$$PrF_0 > -12/5\alpha^2, \tag{30}$$

so that no physically realistic solution is possible if the convective *inflow* is too rapid.

It is interesting to note that (23) and (25) never admit any boundary-layer-like behaviour at the walls for either the velocity or the temperature for any values of F_0 and α and regardless of any limitation such as (30). This is in marked contrast to the constant-density situation for large negative F_0; the appearance of velocity boundary layers in this case is a crucial feature (see also the next section).

Turning to case B, which roughly models a diffusion flame at the centre of the channel, it is easy to show that (19), (22) and (23) give

$$H = H_A - H_0'\sin\theta, \tag{31}$$

where H_A is defined in (25). The various distributions of temperature with respect to θ are therefore simply those discussed above with the addition of a component $-H_0'\sin\theta$. Clearly the energy flux from the 'flame', in the form of the magnitude of H_0' ($\geqslant 0$), is crucial, particularly with respect to its effect on condition (29). The numerator in the quotient on the right-hand side of this inequality must be replaced by $H_0'\tan\alpha - 1$, so that it is possible for inflow to be entirely forbidden if H_0' is too large. Evidently a sufficiently large wall cooling rate can never be achieved if the rate of energy addition from the flame is too large.

In the present circumstances ($n = 1$) the value of HP follows directly from (14a) and (23):

$$HP = \tfrac{1}{3}F_0(7\cos\theta - 4\cos\alpha)/(1-\cos\alpha). \tag{32}$$

We observe that HP is everywhere of the same sign as F_0. Equation (16) shows that the pressure p diminishes in the flow direction like $1/r$ for an outflow ($F_0 > 0$). When $F_0 < 0$, and the flow is inwards, p also diminishes in the flow direction. The result (23) shows that there is no reverse flow (separation), even when F_0 is large and positive, but we have not found a separation-free diffuser since the heating necessary to produce this condition also causes the pressure to fall! The result is unaffected by the presence or absence of a centre-line 'flame'.

The fact that HP in (32) is negative for the case of inflow ($F_0 < 0$) does not necessarily mean that the absolute pressure takes on a meaningless negative value everywhere. The quantity p in this analysis is essentially a relative pressure, which differs from the absolute quantity by a constant which is open for selection; making this constant sufficiently large and positive will confine the region of negative absolute pressures to within an arbitrarily small distance from the origin.

6. A boundary-layer approximation for case II

When the temperature does not vary with radial position ($n = 0$), for any given θ the similitude of case II makes it possible to include a temperature-dependent viscosity. Equations (11) and (15c) show that

$$H^{1+\beta} = (\mu/\mu_0)\,H = a + b\theta \equiv h, \tag{33}$$

where a and b are constants and h is defined for later convenience. If we depart a little from the strict requirements of the case B boundary conditions, and select the wall temperature (i.e. a value for $H(\alpha)$) rather than the centre-line gradient H_0', it is possible to write (33) in the form

$$\mu/\mu_0 = h^{\omega} = \{1-(1-h_\alpha)\,(\theta/\alpha)\}^{\omega}, \tag{34}$$

where

$$\omega = \beta/(1+\beta), \quad h_\alpha = H_\alpha^{1+\beta} < 1 \quad (h_\alpha > 0), \tag{35}$$

$H(0) = 1$ and $H_\alpha \equiv H(\alpha)$. If h_α is allowed to equal unity both T and μ are constants and it is clear that case A boundary conditions lead directly to the constant-density velocity distributions; hence the limitation of h_α in (35).

One can obviously use (33) and (34) in (15d) and produce an equation for F alone; it is not possible to solve this equation analytically and we turn to a boundary-layer approximation which reveals a number of interesting features of an inflow at high Reynolds number in the presence of a centre-line 'flame'.

The reciprocal Reynolds number ϵ is defined in terms of the net mass flux Q [see (20)] by

$$\frac{Q}{\mu_0} = 2\left|\int_0^{\alpha}\frac{\mu(\theta)}{\mu_0}F(\theta)\,d\theta\right| = \frac{2\alpha}{\epsilon}, \tag{36}$$

where use has been made of the symmetry condition. Defining appropriately scaled radial-mass-flux and pressure variables e and s, respectively, where

$$e = \epsilon(\mu/\mu_0)\,F, \quad s = \epsilon^2(\mu/\mu_0)^2\,P, \tag{37), (38}$$

equations (15a, b) can be rewritten in the form

$$\epsilon h^{-1+\omega}\{h^{\omega}(eh^{1-\omega})'\}' + 2s = -e^2, \tag{39}$$

$$-\epsilon 2(eh)' + (sh^{1-\omega})' = 0. \tag{40}$$

The variable e is evidently normalized by the condition [see (36) and (37)]

$$\frac{1}{\alpha}\left|\int_0^{\alpha} e(\theta)\,d\theta\right| = 1. \tag{41}$$

In the limit $\epsilon \to 0$ the mass flux Q becomes large and (39) and (40) look properly set to exhibit a boundary-layer type of solution. There is at present no restriction on the sign of $e(\theta)$; the tacit presumption that e is $O(1)$ as $\epsilon \to 0$ follows from (41) with the implied restriction that it is a reasonably smooth quantity that is likely to be of one sign only. Then s can likewise be presumed to be $O(1)$ and h is evidently so [see (34)]. The only boundary condition that we require is the no-slip condition

$$e(\alpha) = 0, \tag{42}$$

since we imply symmetry throughout, and the case B specification of F_0 is replaced by the integral condition (41).

An outer solution of (39) and (40) which makes

$$\psi(\theta;\epsilon) \sim \psi_1(\theta), \quad \psi = e, s, \quad \text{as} \quad \epsilon \to 0 \quad \text{with} \quad \theta \quad \text{fixed} \tag{43}$$

gives

$$s_1 h^{1-\omega} = -A, \tag{44}$$

where A is a positive constant; the latter follows from (39), since $h > 0$ and

$$-e_1^2 = 2s_1 = -2Ah^{-1+\omega}. \tag{45}$$

The value of A is found from (41) to be given by

$$(2A)^{\frac{1}{2}} = \tfrac{1}{2}(1+\omega)(1-h_\alpha)(1-h_\alpha^{\frac{1}{2}(1+\omega)})^{-1}, \tag{46}$$

so that e_1 and s_1 can now be found from (45) and (46).

Condition (42) requires $e(\alpha)$ to vanish, which, since

$$e_1(\alpha) = \pm(2A)^{\frac{1}{2}} h_\alpha^{-\frac{1}{2}(1-\omega)} = \pm\tfrac{1}{2}(1+\omega)(1-h_\alpha)(h_\alpha^{\frac{1}{2}(1-\omega)} - h_\alpha)^{-1} \neq 0, \tag{47}$$

it evidently does not do if $e(\theta;\epsilon)$ is represented by $e_1(\theta)$. The remedy for this deficiency in $e_1(\theta)$ is the insertion of a boundary layer adjacent to the walls $\theta = \pm\alpha$. We need deal with only the upper half of the channel; we define

$$\Theta = (\alpha - \theta)/\sqrt{\epsilon} \tag{48}$$

and construct inner solutions via

$$\psi(\theta;\epsilon) \sim \Psi_1(\Theta) \quad \text{as} \quad \epsilon \to 0 \quad \text{with} \quad \Theta \quad \text{fixed}, \tag{49}$$

where $\Psi = E, S$ when $\psi = e, s$. It follows that

$$-E_1^2 = 2S_1 + h_\alpha^\omega E_1'', \tag{50}$$

$$\text{constant} = S_1 h_\alpha^{1-\omega}. \tag{51}$$

Matching the solutions for s shows that

$$S_1 = -Ah_\alpha^{-1+\omega} \tag{52}$$

and defining x and $y(x)$ as

$$x = (2A)^{\frac{1}{4}} h_\alpha^{-\frac{1}{4}(1+\omega)}\,\Theta, \quad -y(x) = h_\alpha^{\frac{1}{2}(1-\omega)}(2A)^{-\frac{1}{2}} E_1 \tag{53}$$

leads to

$$-y'' + y^2 - 1 = 0, \quad y(0) = 0, \quad y(\infty) = \mp 1. \tag{54}$$

The first condition on y follows from the no-slip requirement and the second is the matching condition, which makes use of (47) amongst other things.

Writing $y' = w$ transforms (54) into an autonomous first-order equation, which shows (Cole 1968, p. 149) that there is *no* solution of the present problem with $y(\infty) = -1$. Alternatively, if $y(\infty) = -1$ the differential equation in (54) requires $-\frac{1}{2}y'^2 + \frac{1}{3}y^3 - y = +\frac{2}{3}$, and this makes it impossible for y to be zero anywhere. Hence, just as in the constant-density case (Rosenhead 1963, p. 150), there are no boundary-layer solutions for the outflow problem.

For an inflow situation (54) with $y(\infty) = +1$ has the solution

$$y = 3\tanh^2\{(x/\sqrt{2})+\varphi\}-2, \quad \varphi = \ln(\sqrt{3}+\sqrt{2}), \tag{55}$$

which exhibits the wall boundary-layer character of E_1 and hence, via (49), that of e, directly.

Examining the situation at the centre-line of the channel we observe that, since

$$\mu FH = (\mu_0/\epsilon)\, eh^{1-\omega}, \tag{56}$$

the solution which makes $e \sim e_1 = (2A)^{\frac{1}{2}} h^{-\frac{1}{2}(1-\omega)}$ [see (45)] does not satisfy the condition that $(\mu FH)'$ must vanish there, since $h'(0) \neq 0$ by hypothesis. It is in fact necessary to insert a further velocity boundary layer around the flame sheet in order to acquire the required continuity of the rate-of-strain tensor. Writing

$$e(\theta;\epsilon) = -(2A)^{\frac{1}{2}} + \epsilon^{\frac{1}{2}}\mathscr{E}(\phi;\epsilon), \quad \phi = \theta/\epsilon^{\frac{1}{2}}, \tag{57}$$

it follows from (39), (40) and matching with solutions (44) and (45) that the new function $\mathscr{E}$ is adequately represented by $\mathscr{E}_1(\phi)$ to first order in ϵ, where $\mathscr{E}_1$ satisfies

$$\mathscr{E}_1'' + 2(2A)^{\frac{1}{2}} e_1'(0)\,\phi = 2(2A)^{\frac{1}{2}}\mathscr{E}_1, \tag{58}$$

$$\mathscr{E}_1'(0) = 2e_1'(0), \quad \mathscr{E}(\phi\to\infty)\to e_1'(0)\,\phi, \quad e_1'(0) = -(2A)^{\frac{1}{2}}(1-\omega)(1-h_\alpha)/2\alpha. \tag{59}$$

Equation (40) makes $sh^{1-\omega}$ constant to first order and matching makes the constant equal to $-A$, as in (44); this result is used to produce the second term in (58).

The first boundary condition in (59) derives from (56) and the requirement

$$(\mu FH)' = 0$$

at $\theta = 0 = \phi$; this exact condition is equivalent to

$$e'(0) = (1-\omega)(1-h_\alpha)\, e(0)/\alpha,$$

which translates into the quoted condition under the appropriate new limit, namely $\epsilon \to 0$, ϕ fixed. The solution for $\mathscr{E}_1$ is

$$\mathscr{E}_1 = e_1'(0)\,\phi - \frac{e_1'(0)}{[2(2A)^{\frac{1}{2}}]^{\frac{1}{2}}}\exp\{-[2(2A)^{\frac{1}{2}}]^{\frac{1}{2}}\phi\}. \tag{60}$$

We can now see how the temperature-linked density and viscosity variations affect the flow in a heated channel. Equations (16), (37) and (38) show that

$$u = \left(\frac{\mu_0}{\epsilon\rho_0 r}\right) eh^{1-\omega}, \quad p = \left(\frac{\mu_0}{\epsilon\rho_0 r}\right)^2 sh^{1-\omega}, \tag{61}$$

so that in the centre of the channel, away from both the walls and the 'flame', where (43) provides a valid solution, we have

$$u \sim -\left(\frac{\mu_0}{\epsilon\rho_0 r}\right)(2A)^{\frac{1}{2}} h^{\frac{1}{2}(1-\omega)}, \quad p \sim -\left(\frac{\mu_0}{\epsilon\rho_0 r}\right)^2 A. \tag{62}$$

The pressure is therefore constant across the channel in this region, at least to first order, while the gas velocity varies with θ in a way which is intimately connected with the temperature variations. Observing that $0 \leqslant \omega < 1$ for $0 \leqslant \beta < \infty$, (57) shows that u varies rather less with θ as β, and hence ω, increases, but, while this dependence on the viscosity-temperature index is interesting, it is less significant than the fact that a constant-density model would predict no variations of u with θ at all in these regions of the flow. This provides some direct evidence for the view, expressed in §3, that density variations with temperature are more important than those of the viscosity. The effects of h_α, or H_α, and ω on the central portions of the channel flow are necessarily reflected in the details of the wall boundary layer, as can be seen from (53) and (55); evidently the thickness and velocity amplitude of the layer depend upon h_α and ω in quite a complicated fashion but, since (55) is also the dimensionless velocity profile for the constant-density problem, the general *shape* of the variations is not dependent upon these quantities.

Evidently the 'flame's' velocity boundary layer has a much smaller velocity amplitude than is to be found at the walls [consult (49) and (57)] and its strong dependence on the temperature–viscosity index β is interesting [(60) shows that $\mathscr{E}_1 \propto e_1'(0)$ and (59) that $e_1'(0) \propto 1-\omega$; then (35) shows that $\mathscr{E}_1 \propto 1/(1+\beta)$]. Since $e_1'(0)$ is essentially negative it follows from (57) and (60) that the 'flame' slows the flow down slightly in its immediate neighbourhood.

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